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Deformed coherent and squeezed states of multiparticle processes

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Abstract. Deformed squeezed states are introduced as the q-analogues of the conventional undeformed harmonic oscillator algebra squeezed states. It is shown that the boundary vectors in the matrix-product states approach to multiparticle diffusion processes are deformed coherent or squeezed states of a deformed harmonic oscillator algebra. A deformed squeezed and coherent-states solution to the n-species stochastic diffusion boundary problem is proposed and studied.

1 Introduction

Coherent states have a wide range of applications to various problems in many different areas of physics. Introduced by Schrödinger [1] in the early days of quantum mechanics, the harmonic oscillator coherent states were developed for the first time by Glauber for quantized electromagnetic radiation [2]. They were generated by the action of the displacement operator on the ground state, or equivalently defined as eigenstates of the annihilation operator, and turned out to be orbits of the Weyl–Heisenberg group. This important property led to group-theoretical generalizations by Perelomov [3] and Gilmore [4] for an arbitrary Lie group and to the formulation of coherent states as orbits of the group with respect to a stationary subgroup.

With the invention of quantum groups and the hopes that rich non-commutative structures will amount to new results in field theory and statistical physics, generalized coherent states [5,6] for the deformed Heisenberg algebra and for compact quantum groups [6] were introduced and studied.

Coherent states exhibit two basic characteristics, namely continuity and resolution of unity, which are the minimum requirements for a set of vectors to be referred to as generalized coherent states. According to Klauder [7], a coherent state $|l\rangle$ where the (complex) label l is an element of an appropriate label space $\mathcal{L}$, endowed with the notion of topology, is a vector of a Hilbert space $\mathcal{H}$ such that
(i) the vector $|l\rangle$ is strongly continuous in the label l,
(ii) there exists a positive measure δl on $\mathcal{L}$ so that the unit operator on $\mathcal{H}$ admits a resolution of unity $I = \int |l\rangle\langle l|\delta l$.

Consequently any quantum state $|\psi\rangle$ can be represented by its projections onto the different coherent states $\psi(l) = \langle l|\psi\rangle$, and similarly any operator A can be represented by its coherent-states matrix elements $\frac{\langle l|A|l'\rangle}{\langle l|l'\rangle}$.

By their origin the coherent states are quantum states, but at the same time they are parametrized by points in the phase space of a classical system. This makes them very suitable for the study of systems where one encounters a relationship between classical and quantum descriptions. From this point of view, interacting many-particle systems with stochastic dynamics provide an appropriate playground to enhance the utility of generalized coherent states.

A stochastic process is described in terms of a master equation for the probability distribution $P(s_i, t)$ of a stochastic variable $s_i = 0, 1, 2, \ldots, n-1$ at a site $i = 1, 2, \ldots, L$ of a linear chain. A configuration on the lattice at a time t is determined by the set of occupation numbers $s_1, s_2, \ldots, s_L$ and a transition to another configuration s' during an infinitesimal time step $\mathrm{d}t$ is given by the probability $\Gamma(s, s')\mathrm{d}t$. The time evolution of the stochastic system is governed by the master equation

$$\frac{\mathrm{d}P(s,t)}{\mathrm{d}t} = \sum_{s'} \Gamma(s,s')P(s',t) \tag{1}$$

for the probability $P(s,t)$ of finding the configuration s at a time t. With the restriction of dynamics that changes of configuration can only occur at two adjacent sites, the rates for such changes depend only on these sites. The two-site rates $\Gamma \equiv \Gamma^{ik}_{jl}$, $i, j, k, l = 0, 1, 2, \ldots, n-1$ are assumed to be independent of the position in the bulk. At the boundaries, i.e. sites 1 and L, additional processes can take place with single-site rates L^i_k and R^i_k, $i, k = 0, 1, \ldots, n-1$. For processes where each lattice site can be occupied by a finite number of different-type particles, the master equation can be mapped to a Schrödinger equation in imaginary time of an n-state quantum spin-S Hamiltonian ($n = 2S+1$ distinct states) with nearest-neighbor interaction in the bulk and single-site boundary terms

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t} = -HP(t), \qquad H = \sum_j H_{j,j+1} + H^{(\mathrm{L})} + H^{(\mathrm{R})}. \tag{2}$$

The probability distribution thus becomes a state vector in the configuration space of the quantum spin chain and the ground state of the Hamiltonian, in general non-Hermitian, corresponds to the steady state of the stochastic dynamics where all probabilities are stationary.

The mapping provides a connection with integrable quantum spin chains and allows for exact results of the stochastic dynamics with the formalism of quantum mechanics.

A different description, which is also based on the relationship of a Markov process probability distribution with the quantum Hamiltonian picture, is the matrix-product states approach to stochastic dynamics [8, 9]. The idea is that the stationary probability distribution, i.e. the ground state of a quantum Hamiltonian with nearest-neighbor interaction in the bulk and single-site boundary terms, can be expressed as a product of (or a trace over) matrices that form a representation of a quadratic algebra

$$\Gamma^{ik}_{jl} D_i D_k = x_l D_j - x_j D_l, \qquad i,j,k,l = 0,1,\ldots,n-1, \tag{3}$$

determined by the dynamics of the process. For diffusion processes that will be considered in this paper, $\Gamma^{ik}_{ki} = g_{ik}$ and the n-species diffusion quadratic algebra has the form

$$g_{ik} D_i D_k - g_{ki} D_k D_i = x_k D_i - x_i D_k, \tag{4}$$

where g_{ik} and g_{ki} are positive (or zero) probability rates, x_i are c-numbers and $i,k = 0,1,\ldots,n-1$. (No summation over repeated indices in (4).) The algebra has a Fock representation in an auxiliary Hilbert space where the n generators D act as operators. For systems with periodic boundary conditions, the stationary probability distribution is related to the expression

$$P(s_1,\ldots,s_L) = \mathrm{Tr}(D_{s_1} D_{s_2} \ldots D_{s_L}). \tag{5}$$

When boundary processes are considered the stationary probability distribution is related to a matrix element in the auxiliary vector space

$$P(s_1,\ldots,s_L) = \langle w | D_{s_1} D_{s_2} \ldots D_{s_L} | v \rangle \tag{6}$$

with respect to the vectors $|v\rangle$ and $\langle w|$, determined by the boundary conditions

$$\langle w | (L^k_i D_k + x_i) = 0, \qquad (R^k_i D_k - x_i)|v\rangle = 0, \tag{7}$$

where the x-numbers sum up to zero, because of the form of the boundary rate matrices

$$L^i_i = -\sum_{j=0}^{n-1} L^i_j, \qquad R^i_i = -\sum_{j=0}^{n-1} R^i_j, \qquad \sum_{i=0}^{n-1} x_i = 0. \tag{8}$$

These relations simply mean that one associates with an occupation number s_i at position i a matrix $D_{s_i} = D_k$ ($i = 1,2,\ldots,L$; $k = 0,1,\ldots,n-1$) if a site i is occupied by a k-type particle. The number of all possible configurations of an n-species stochastic system on a chain of L sites is n^L and this is the dimension in the configuration space of the stationary probability distribution as a state vector. Each component of this vector, i.e. the (unnormalized) steady-state weight of a given configuration, is a trace or an expectation value in the auxiliary space given by (5) or (6). The quadratic algebra reduces the number of independent components to only monomials symmetrized upon using the relations (4).

The algebra (4) admits an involution through the mapping $D_i \to D_i^+$, ($D_i \to -D_i^+$) and $g^+_{ik} = -g_{ki}$ ($g^+_{ik} = g_{ki}$) for real parameters $x_i = \bar{x}_i$.

Relations (4) allow an ordering of the elements D_k and, in order to find the stationary probability distribution, one has to compute traces or matrix elements with respect to the vectors $|v\rangle$ and $\langle w|$ of ordered monomials of the form

$$D^{m_1}_{s_1} D^{m_2}_{s_2} \ldots D^{m_l}_{s_l}, \tag{9}$$

where $s_1 < s_2 < \ldots < s_l$, $l \geq 1$ and $m_1, m_2, \ldots, m_l$ are non-negative integers. Monomials of given order are the Poincaré–Birkhoff–Witt (PBW) basis for polynomials of fixed degree, as is the stationary probability distribution. The n elements D_k obeying the $n(n-1)/2$ relations (4) generate an associative algebra with an unit e for which the ordered monomials (9) form a linear basis, the PBW basis.

In the known example of exactly soluble 2- and 3-species models, through the matrix-product ansatz, the solution of the quadratic algebra is provided by a deformed bosonic oscillator algebra, if both g_{ik} and g_{ki} differ from zero, or by infinite-dimensional matrices, if $g_{ik} = 0$. In the general n case, because of the ordering procedure, the solution of the quadratic algebra has to be consistent with the diamond lemma in ring theory, also known as the braid associativity condition in quantum groups. As shown in [10, 11], if all parameters x_i are equal to zero on the RHS of (4), the homogeneous quadratic algebra defines a multiparameter quantized non-commutative space realized equivalently as a q-deformed Heisenberg algebra [12, 13] of n oscillators depending on $n(n-1)/2+1$ parameters (in general on $n(n-1)/2+n$ parameters):

$$\begin{aligned} a_i a_i^+ - r_i a_i^+ a_i &= 1, \\ a_i^+ a_j^+ - q_{ji} a_j^+ a_i^+ &= 0, \\ a_i a_j - q_{ji} a_j a_i &= 0, \\ a_i a_j^+ - q_{ji}^{-1} a_j^+ a_i &= 0, \end{aligned} \tag{10}$$

where $i < j$; $i,j = 0,1,\ldots,n-1$, and the deformation parameters r_i, q_{ij} are model-dependent parameters given in terms of the probability rates, and the associative algebra generated by the elements D_i in this case belongs to the universal enveloping algebra of the multiparameter deformed Heisenberg algebra. For a non-homogeneous algebra with x-terms on the RHS of (4), only then is braid associativity satisfied if, out of the coefficients x_i, x_k, x_l corresponding to an ordered triple $D_i D_k D_l$, either one coefficient x is zero or two coefficients x are zero, and the rates are respectively related. The diffusion algebras in this case can be obtained by either a change of basis in the n-dimensional non-commutative space or by a suitable change of basis of the lower-dimensional quantum space realized equivalently as a lower-dimensional deformed Heisenberg algebra. The appearance of the nonzero linear terms in the RHS of the quantum plane relations leads to a lower-dimensional non-commutative space.

Proposition I. The boundary vectors with respect to which one determines the stationary probability distribution of the n-species diffusion process are generalized, coherent or squeezed states of the deformed Heisenberg algebra underlying the algebraic solution of the corresponding quadratic algebra.

This paper is organized as follows. We first review the basic properties of the deformed oscillator coherent states that are known in the literature. We then define a deformed squeezed state of a pair of deformed oscillators by analogy with the conventional squeezed states as the eigenstate of the deformed boson operators' linear combination and study their squeezing properties. Such a q-generalization of the conventional undeformed squeezed states is not known. As a physical application of the deformed coherent and the considered squeezed states we propose and study the boundary problem solution of the general n-species stochastic diffusion process. We argue that, depending on the boundary conditions, the boundary vectors are either the deformed boson operator coherent or the suggested deformed squeezed states. We finally comment on the two-species simple exclusion process to serve as an example of an unified description in terms of coherent states of both the partially and the fully asymmetric processes.

2 Coherent states of a q-deformed Heisenberg algebra

The conventional harmonic oscillator coherent states are defined either

(i) directly by the action of the displacement operator $D(z) = \exp(za^+ - \bar{z}a)$ on the vacuum, or

(ii) as an eigenstate of the annihillation operator a. It is the displacement operator method that best reveals the group geometric properties of the coherent states; it allowed for generalization to arbitrary Lie groups, but it turned out not to work successfully for quantum groups with conventional complex variables z. The generalization to the deformed boson case went along the annihilation operator method and we review here the main lines of the known results [5].

We consider an associative algebra with generators a, a^+ and $q^{\pm N}$ with the defining relations

$$aa^+ - qa^+a = 1, \quad q^N a^+ = qa^+ q^N, \quad q^N a = q^{-1} a q^N, \tag{11}$$

where $0 < q < 1$ is a real parameter and

$$a^+ a = \frac{1 - q^N}{1 - q} \equiv [N]. \tag{12}$$

A Fock representation is obtained in a Hilbert space spanned by the orthonormal basis $\frac{(a^+)^n}{\sqrt{[n]!}}|0\rangle = |n\rangle, n = 0, 1, 2, \ldots$ and $\langle n|n'\rangle = \delta_{nn'}$:

$$a|0\rangle = 0, \quad a|n\rangle = [n]^{1/2}|n-1\rangle, \quad a^+|n\rangle = [n+1]^{1/2}|n+1\rangle. \tag{13}$$

The Hilbert space consists of all elements $|f\rangle = \sum_{n=0}^{\infty} f_n|n\rangle$ with complex f_n and finite norm with respect to the scalar product $\langle f|f\rangle = \sum_{n=0}^{\infty} |f_n|^2$. The q-deformed oscillator algebra has a Bargmann–Fock representation on the Hilbert space of entire analytic functions.

Generalized or q-deformed coherent states are defined as the eigenstates of the deformed annihilation operator a and are labelled by a continuous (in general complex) variable z:

$$a|z\rangle = z|z\rangle, \qquad |z\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]!}}|n\rangle. \tag{14}$$

These vectors belong to the Hilbert space for $|z|^2 < [\infty] = \frac{1}{1-q}$.

The scalar product of two coherent states for different values of the parameter z is non-vanishing:

$$\langle z|z'\rangle = \sum_{0}^{\infty} \frac{(\bar{z}z')^n}{[n]!} = e_q^{\bar{z}z'}, \tag{15}$$

and they can be properly normalized with the help of the q-exponent on the RHS of (15):

$$|z\rangle = \exp_q\left(-\frac{|z|^2}{2}\right) \exp_q(za^+)|0\rangle. \tag{16}$$

The q-deformed coherent states reduce to the conventional coherent states of a one-dimensional Heisenberg algebra in the limit $q \to 1^-$. These generalized coherent states carry the basic characteristics of the conventional ones, namely continuity and completeness (resolution of unity). We briefly sketch the main properties as they were analyzed in relation to the deformed algebra representation on the Hilbert space of entire analytic functions. The representation space is spanned by the orthonormal basis of polynomials $u_n = \frac{z^n}{\sqrt{[n]!}}$, $n = 0, 1, 2, \ldots$ and a scalar product of two elements $g(z)$ and $f(z)$ is given by

$$\langle g|f\rangle = \int \bar{g}(z) f(z) \exp_q(-\bar{z}z) \mathrm{d}_q^2 z. \tag{17}$$

The integration in (17) over the complex variable $z = |z|\exp(\mathrm{i}\phi)$ is performed as

$$\int \mathrm{d}_q^2 z = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi \int_0^{\infty} \mathrm{d}_q |z|^2, \tag{18}$$

and the Jackson q-integral of a function $F(x)$ of a real variable x is defined as the inverse of the q-derivative D_q

$$\int_0^\infty F(x)\mathrm{d}_q x = (1-q)x\sum_{l=0}^{\infty} q^l F(q^l x),$$

$$D_q f(z) = \frac{f(z)-f(qz)}{z-qz}. \quad (19)$$

The scalar product in (17) allows for a resolution of the unit operator:

$$I = \int |z\rangle\langle z| \exp_q(-|z|^2)\mathrm{d}_q^2 z. \quad (20)$$

Using the completeness relation one can expand any state $|f\rangle$ in the coherent states

$$|f\rangle = \int \mathrm{d}_q^2 z |z\rangle \exp_q(-|z|^2)\langle \bar{z}|f\rangle, \quad (21)$$

where the function $f(z) = \langle \bar{z}|f\rangle$ determines the state completely and is called the symbol of the state. The completeness relation gives rise to a functional representation of operators as well

$$(Af)(z) = \int A(z,z')f(z')\exp_q(\bar{z}'z)\exp_q(-\bar{z}'z')\mathrm{d}_q^2 z', \quad (22)$$

$A(z,z') = \frac{\langle z|A|z'\rangle}{\langle z|z'\rangle}$ being the covariant symbol of the operator A. The trace of the operator A is given by $\mathrm{Tr}A = \int \mathrm{d}_q^2 z \exp_q(-|z|^2)\langle z|A|z\rangle$. One thus has

$$\langle \bar{z}|a^+|f\rangle = zf(z), \quad (23)$$

$$\langle \bar{z}|a|f\rangle = D_q f(z),$$

$$\langle \bar{z}|N|f\rangle = z\frac{\mathrm{d}}{\mathrm{d}z}f(z),$$

which is the Bargmann–Fock representation of the deformed oscillators and number operator.

As shown in [5], in the limit $q \to 1^-$, the Bargmann–Segal representation space of the undeformed algebra on a Hilbert space of entire functions is obtained, while in the limit $q \to 0$, the Hilbert space becomes the Hardy–Lebesgue space of functions on the circle $|z| = 1$.

3 Squeezed states of a deformed oscillator algebra

The conventional squeezed states [14] for the harmonic oscillator operators are obtained directly from a conventional coherent state $|z\rangle$ by applying the squeezed operator $S(\xi)$:

$$S(\xi)D(z)|0\rangle = |z,\xi\rangle, \quad (24)$$

where the unitary operator $S(\xi)$ depends on a (complex) parameter ζ. The squeezed state is an eigenstate of the transformed annihilation operator a,

$$S(\xi)aS^+(\xi) = A(a^+,a), \quad (25)$$

so that one has

$$A(a^+,a)|z,\xi\rangle = z|z,\xi\rangle. \quad (26)$$

The unitary transformation leaves the commutator $[a,a^+]$ invariant and can be realized as a linear canonical transformation

$$A(a^+,a) = \mu a + \nu a^+, \quad (27)$$

with $|\mu|^2 - |\nu|^2 = 1$. The unitary operator that leads to such a linear transform has the form

$$S(\xi) = \exp\frac{1}{2}(\xi(a^+)^2 - \bar{\xi}a^2), \quad (28)$$

and with real $\xi = s$ one has $A = a\cosh s + a^+\sinh s$. The squeezed states are also equivalently defined as coherent states of the group $SU(1,1)$ [3] by the action of the raising operator $K^+ = \frac{1}{2}(a^+)^2$ on the vacuum:

$$|s\rangle = \exp\frac{1}{2}s(a^+)^2|0\rangle. \quad (29)$$

Attempts at generalizing these definitions to the case of deformed oscillators have not yet been quite successful. As for the first definition there were discussions and argumentations that a squeezed operator (as well as a displacement operator $D(z)$) can be consistently defined [15] assuming the variables z, ξ to be non-commuting. On the other hand, the generalization of the second definition to the deformed case gives a state $\exp_q \frac{1}{[2]}\xi(a^+)^2|0\rangle$ that is not normalizable [16]. However, since the action of the (conventional) unitary squeezed operator results in a linear transformation on the oscillators, we are lead by this idea to keep the linear structure of the deformed squeezing operator and assume an analogous definition.

Proposition IIa. Let a and a^+ generate a deformed Heisenberg algebra in the equivalent commutator form of a defining relation

$$[a,a^+] = q^N, \quad (30)$$

Then there is a two-parameter-dependent linear map to a pair of "quasi"-oscillators with (the symbol of) a "quasi-particle" number operator $\mathcal{N}$

$$A = \mu a + \nu a^+, \qquad A^+ = \bar{\mu}a^+ + \bar{\nu}a. \quad (31)$$

These operators generate a deformed Heisenberg algebra

$$[A,A^+] = q^{\mathcal{N}}, \quad (32)$$

provided

$$q^{\mathcal{N}} = (|\mu|^2 - |\nu|^2)q^N. \quad (33)$$

In the limit $q \to 1^-$ the relation between the parameters of the conventional squeezed state is recovered [17]. In the deformed "quasi"-oscillator algebra Fock representation space with a vacuum $|0\rangle_s$ one can define a normalizable coherent state $|\zeta\rangle_s$ as the eigenvector of the annihilation operator A:

$$|\zeta\rangle_s = e_q^{-\frac{1}{2}|\zeta|^2} e_q^{\zeta A^+}|0\rangle_s. \quad (34)$$

In order to generate a deformed squeezed state directly one needs of course to explicitly construct an operator

$S_q(\mu,\nu)$, the q-analogue of the squeezed operator whose transformation of the oscillators amounts to the linear map in (31), $S_q a S_q^{-1} = A$. This question remains open despite the encouraging fact that the linear transformation has the proper limit $q \to 1^-$.

Proposition IIb. A squeezed state of the deformed creation and annihilation operators is a normalized solution of the eigenvalue equation

$$(\mu a + \nu a^+)|\zeta,\mu,\nu\rangle_s = \zeta|\zeta,\mu,\nu\rangle_s = A|\zeta\rangle_s. \tag{35}$$

This proposition is motivated by the analogy with the non-deformed case and by the fact that such normalized eigenstate vectors of the above linear combination of q-deformed oscillators appear in the solution of the boundary problem of a many-particle non-equilibrium system.

The conventional undeformed squeezed states were originally introduced as states in which one of the variances of the two quadratures of the oscillator operators was smaller than the variance $\frac{1}{2}$ in the Glauber coherent states minimizing the Heisenberg uncertainty relation. A generalization [18–20] to any two Hermitian operators x, p defines a squeezed state as

(i) a state in which one of the two variances $(\delta x)^2$ or $(\delta p)^2$ is smaller than the modulus half of the commutator mean value in this state, $\frac{1}{2}|\langle[x,p]\rangle|$, or via the minimum-uncertainty method as

(ii) a state in which one of the two variances is smaller than their common minimal value δ^2_{min} for which the equality $(\delta x)^2 = (\delta p)^2 = \frac{1}{2}|\langle[x,p]\rangle|$ holds. Eigenstates of linear x–p combinations exhibit strong squeezing properties depending only on the parameters involved in the combination. Upon adjusting these parameters one can achieve that either variance could reach the limit value zero.

Even though in the proposed definition (34) and (35) of the squeezed state as an eigenstate $|\zeta\rangle_s$ of the annihilation operator A, the μ,ν dependence seems to be suppressed, such states exhibit squeezing properties similar to the conventional ones. To show this we consider the Hermitian quadrature operators

$$x = \frac{1}{\sqrt{2}}(a + a^+), \qquad p = \frac{1}{\mathrm{i}\sqrt{2}}(a - a^+), \tag{36}$$

where the boson operators obey the relation of the form (30) and consequently the operators x and p satisfy the deformed canonical commutation relation

$$[x,p] = \mathrm{i}q^N. \tag{37}$$

The number operator N in the Fock space boson oscillator representation is positive definite and Hermitian. Hence for $0 < q < 1$ the deformed operator q^N is Hermitian too. Any two Hermitian operators obeying a commutation relation of the type (37) satisfy a generalized Heisenberg–Robertson inequality of the form

$$(\delta x)^2(\delta p)^2 \geq \frac{1}{4}|\langle[x,p]\rangle|^2, \tag{38}$$

where $(\delta x)^2 = \langle(x - \langle x\rangle)^2\rangle$ and similary $(\delta p)^2 = \langle(p - \langle p\rangle)^2\rangle$ are the variances in any state. This is the accepted form of a deformed uncertainty relation for the two quadratures x and p. In the limit $q \to 1^-$ the commutator (37) is a c-number and the generalized inequality (38) becomes the celebrated Heisenberg uncertainty relation $(\delta x)^2(\delta p)^2 \geq \frac{1}{4}$ of quantum physics. Making use of the expressions (36) for x and p in terms of a and a^+, we find the uncertainties

$$\begin{aligned}(\delta x)^2 &= \langle x^2\rangle - \langle x\rangle^2 = u_1 + u_2, \\ (\delta p)^2 &= \langle p^2\rangle - \langle p\rangle^2 = u_1 - u_2,\end{aligned} \tag{39}$$

where

$$\begin{aligned}u_1 &= \frac{1}{2}\langle a^+a + aa^+\rangle - \langle a\rangle\langle a^+\rangle, \\ u_2 &= \langle a^2\rangle - \langle a\rangle^2 + \langle(a^+)^2\rangle - \langle a^+\rangle^2.\end{aligned} \tag{40}$$

If one calculates now the mean values in u_1 and u_2 with respect to the deformed coherent states $|z\rangle$ of the oscillators a and a^+, one finds that $u_2 = 0$, i.e. the deformed uncertainties are equal:

$$(\delta x)^2 = (\delta p)^2 = \frac{1}{2}\exp_q\left((q-1)|z|^2\right). \tag{41}$$

This equality of the deformed x,p uncertainties has been discussed in previous works (see [21] for details). We comment on it here just because it is important for the discussion of the squeezing properties of the states (35). Equation (41) defines the equal uncertainties $(\delta x)^2$, $(\delta p)^2$ as a function of the positive variables q and $|z|^2$. For any $0 < q < 1$ and for any $0 < |z|^2 < \frac{1}{1-q}$ this function is bounded by its value with respect to the vacuum ($z = 0$)

$$\frac{1}{2}\exp_q\left((q-1)|z|^2\right) < \frac{1}{2}. \tag{42}$$

In the limit $|z|^2 \to \frac{1}{1-q}$ the behavior of the function on the LHS of (42) does not change the validity of this inequality. It reads explicitly

$$\lim_{|z|^2 \to \frac{1}{1-q}} \frac{1}{2}\exp_q\left((q-1)|z|^2\right) = \frac{1}{2}e_q^{-1} < \frac{1}{2}, \tag{43}$$

where $e_q^{-1} = \frac{1}{(1+(1-q))_q^\infty}$. The inequality (42) remains valid also in the limit $q \to 0$ when $|z|^2 < 1$ and one has

$$\lim_{q\to 0} \frac{1}{2}\exp_q\left((q-1)|z|^2\right) = \frac{1}{2}\frac{1}{1+|z|^2} < \frac{1}{2}. \tag{44}$$

According to (41) the deformed coherent states are states of equal uncertainties only. Minimum uncertainty states are labeled by the value z for which the q-exponent in (41) has a minimum. In the limit $q \to 1^-$ for $0 < |z|^2 < \infty$ the equality of the undeformed uncertainties in the Glauber coherent states is recovered. Thus any deformed coherent state $|z,q\rangle$ for $0 \leq q < 1$ and $0 < |z|^2 < \frac{1}{1-q}$ is a Robertson intelligent state and is sometimes referred to in the literature as a squeezed state in the sense of weak squeezing [22], i.e. $(\delta x)^2 < \frac{1}{2}$, and simultaneously $(\delta p)^2 < \frac{1}{2}$.

We proceed further with the discussion of the algebraic states $|\zeta,\mu,\nu,q\rangle$ which reveal stronger squeezing properties, generalizing thus the undeformed case. It is our aim first to calculate the uncertainties with respect to these states and to show that, analogously to the conventional case, they are not equal since $u_2 \neq 0$. For this purpose we first write the inverse of the linear map in (31)

$$a = \frac{\bar{\mu}}{|\mu|^2 - |\nu|^2} A - \frac{\nu}{|\mu|^2 - |\nu|^2} A^+, \qquad (45)$$
$$a^+ = \frac{-\bar{\nu}}{|\mu|^2 - |\nu|^2} A + \frac{\mu}{|\mu|^2 - |\nu|^2} A^+,$$

where $|\mu|^2 - |\nu|^2 \neq 0$, being the Jacobian of the linear transformation (31). The next step is to calculate the quantities u_1 and u_2 with respect to the coherent states of the bosonic pair A, A^+ in the Hilbert space of the Bargmann–Fock representation of these operators.

Exploring the eigenvalue properties of the normalized coherent eigenstates of A, we simply have

$$\langle\zeta|A|\zeta\rangle_s = \zeta, \qquad (46)$$
$$\langle\zeta|A^+|\zeta\rangle_s = \bar{\zeta},$$
$$\langle\zeta|q^{\mathcal{N}}|\zeta\rangle_s = \langle\zeta|q^{\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta}}|\zeta\rangle_s = e_q^{(q-1)|\zeta|^2}.$$

To find the mean values in $u_{1,2}$ with respect to $|\zeta\rangle_s$, we use the expressions for a, a^+ in terms of A, A^+ according to (45) and with the help of (46) we obtain

$$2u_1 = \frac{\bar{\mu}\mu + \bar{\nu}\nu}{(|\mu|^2 - |\nu|^2)^2} \exp_q((q-1)|\zeta|^2), \qquad (47)$$
$$2u_2 = \frac{-\bar{\mu}\nu - \bar{\nu}\mu}{(|\mu|^2 - |\nu|^2)^2} \exp_q((q-1)|\zeta|^2).$$

Since $u_2 \neq 0$, this yields a non-equality of the q-deformed uncertainties, which read explicitly

$$(\delta x)^2 = \frac{1}{2} \frac{|\mu - \nu|^2}{(|\mu|^2 - |\nu|^2)^2} \exp_q\left((q-1)|\zeta|^2\right), \qquad (48)$$
$$(\delta p)^2 = \frac{1}{2} \frac{|\mu + \nu|^2}{(|\mu|^2 - |\nu|^2)^2} \exp_q\left((q-1)|\zeta|^2\right).$$

As seen on the RHS of (48) $(\delta x)^2$ as well as $(\delta p)^2$ are up to a corresponding μ-, ν-dependent factor equal to the commutator (positive) mean value $\langle[A, A^+]\rangle$ in the squeezed coherent states $|\zeta\rangle_s$, the eigenstates of the operator A. Since the two pairs of deformed oscillators A, A^+ and a, a^+ are related by the linear transformation (31) the validity of relation (33) leaves the commutator $\mathrm{i}[x,p] = [a, a^+]$ invariant and we thus have (omitting from now on the subscript s of the deformed state $|\zeta\rangle_s$)

$$(\delta x)^2 = \frac{1}{2} \frac{|\mu - \nu|^2}{|\mu|^2 - |\nu|^2} |\langle\zeta|[x,p]|\zeta\rangle|, \qquad (49)$$
$$(\delta p)^2 = \frac{1}{2} \frac{|\mu + \nu|^2}{|\mu|^2 - |\nu|^2} |\langle\zeta|[x,p]|\zeta\rangle|.$$

The deformed eigenstates $|\zeta\rangle$ of the deformed boson operators' linear combination depend, in addition to the real positive deformation parameter q, on the complex parameters ζ, μ, ν. Only two of the latter (i.e. four real) are independent as a consequence of the eigenvalue equation (35). They can be chosen in such a way that

$$\frac{|\mu - \nu|^2}{|\mu|^2 - |\nu|^2} < 1. \qquad (50)$$

Then it follows from (49) that according to the definition (i) $|\zeta\rangle$ is a squeezed state in which

$$(\delta x)^2 < \frac{1}{2}|\langle\zeta|[x,p]|\zeta\rangle|. \qquad (51)$$

Alternatively if

$$\frac{|\mu + \nu|^2}{|\mu|^2 - |\nu|^2} < 1, \qquad (52)$$

then $(\delta p)^2 < \frac{1}{2}|\langle\zeta|[x,p]|\zeta\rangle|$. However in the expressions (49) for the uncertainties in the eigenstates of the linear combination of the operators x and p half of the modulus of the mean value of the commutator, $\frac{1}{2}|\langle\zeta|[x,p]|\zeta\rangle| = (|\mu|^2 - |\nu|^2)^{-1}\frac{1}{2}\exp_q\left((q-1)|\zeta|^2\right)$, depends also on the parameters μ, ν of the linear combination, so it can be large. We need therefore to make use of the definition (ii) of a squeezed state requiring one of the variances to be smaller than the equal uncertainties common minimal value determined by the equality (41) as the minimum in the variable z of the q-exponential function. From the above analyses of the function $\frac{1}{2}\exp_q\left((q-1)|\zeta|^2\right)$ it follows that the minimum of this function is the finite limit $\frac{1}{2}\frac{1}{(1+(1-q))_q^\infty}$ for $|\zeta|^2 \to \frac{1}{1-q}$. Hence according to the expressions (48) $|\zeta\rangle$ is a squeezed state if either

$$\frac{1}{2} \frac{|\mu - \nu|^2}{(|\mu|^2 - |\nu|^2)^2} \exp_q\left((q-1)|\zeta|^2\right) < \frac{1}{2} \frac{1}{(1+(1-q))_q^\infty}, \qquad (53)$$

which is satisfied provided the parameters (in general complex) μ, ν of the linear transformation (31) are chosen in such a way that

$$0 < \frac{|\mu - \nu|^2}{(|\mu|^2 - |\nu|^2)^2} < \frac{\left(1 + (1-q)^2|\zeta|^2\right)_q^\infty}{(1+(1-q))_q^\infty}, \qquad (54)$$

and thus the criterion

$$(\delta x)^2 < (\delta x)^2_{\min} \qquad (55)$$

holds. The ratio at the very RHS of (54) is the basic hypergeometric series ${}_1\Phi_0\left((1-q)|\zeta|^2; q, (q-1)\right)$. Alternatively from (48)

$$\frac{1}{2} \frac{|\mu + \nu|^2}{(|\mu|^2 - |\nu|^2)^2} \exp_q\left((q-1)^2|\zeta|^2\right) < \frac{1}{2} \frac{1}{(1+(1-q))_q^\infty} \qquad (56)$$

is satisfied if

$$0 < \frac{|\mu + \nu|^2}{(|\mu|^2 - |\nu|^2)^2} < \frac{\left(1 + (1-q)^2|\zeta|^2\right)_q^\infty}{(1+(1-q))_q^\infty}, \qquad (57)$$

which gives

$$(\delta p)^2 < (\delta p)^2_{\min}. \tag{58}$$

For $0 < q < 1$ the values $\mu = \pm\nu$ are not admissible. The inequality (55) (or (58)) together with the condition (54) (or (57)) for the parameters μ, ν, ζ, q define the eigenstates $|\zeta\rangle$ of the linear combination $\mu a + \nu a^+$ of the deformed boson operators as generalized squeezed states. In the limit $q \to 1^-$ the corresponding expressions for the x, p uncertainties with respect to the conventional harmonic oscillator squeezed states [17] are recovered. This analogy with the squeezing properties of the quadratures of the boson creation and annihilation operators justifies, in our opinion, the proposed definition of a q-deformed squeezed state in (35) as a q-generalization of the conventional squeezed states.

In the deformed uncertainty relation (38) the variances of the operators x and p enter. For two Hermitian operators a third second moment, their covariance in any state, is defined by

$$\delta(xp) = \frac{1}{2}\langle xp + px\rangle - \langle x\rangle\langle p\rangle. \tag{59}$$

As can be readily verified the covariance $\delta(xp)$ of the quadratures (36) x and p in the deformed boson oscillator coherent state $|z\rangle$ is equal to zero. If we calculate now the x–p covariance in the deformed states $|\zeta, \mu, \nu\rangle$ we obtain

$$\begin{aligned}\delta xp &= \frac{\mathrm{Im}(\mu\bar{\nu})}{|\mu|^2 - |\nu|^2}|\langle[x,p]\rangle_s| \\ &= \frac{\mathrm{Im}(\mu\bar{\nu})}{(|\mu|^2 - |\nu|^2)^2}\exp_q\left((q-1)|\zeta|^2\right).\end{aligned} \tag{60}$$

As seen from (60) the x–p covariance in the deformed squeezed states for complex μ, ν is not zero. It vanishes in the particular case of real μ, ν.

For Hermitian operators with a non-vanishing covariance the Robertson–Heisenberg uncertainty relation becomes the Schrödinger inequality in any state

$$(\delta x)^2(\delta p)^2 - (\delta xp)^2 \geq \frac{1}{4}|\langle[x,p]\rangle|^2. \tag{61}$$

One can further verify that the three second moments in the deformed squeezed states as given by (48) and (60) satisfy the equality

$$(\delta x)^2(\delta p)^2 - (\delta xp)^2 = \frac{1}{4}|\langle\zeta[x,p]\zeta\rangle|^2. \tag{62}$$

The q-deformed squeezed states $|\zeta, \mu, \nu\rangle$ thus minimize the Schrödinger uncertainty relation for the deformed quadratures and are, in fact, generalized Schrödinger intelligent states [20].

4 Physical applications

We consider a diffusion process with n species on a chain of L sites with nearest-neighbor interaction with exclusion, i.e. a site can be either empty or occupied by a particle of a given type. In the set of occupation numbers $(s_1, s_2, \ldots, s_L)$ specifying a configuration of the system $s_i = 0$ if a site i is empty, $s_i = 1$ if there is a first-type particle at a site $i, \ldots, s_i = n-1$ if there is an $(n-1)$th-type particle at a site i. On successive sites the species i and k exchange places with probability $g_{ik}\mathrm{d}t$, where $i, k = 0, 1, 2, \ldots, n-1$. With $i < k$, g_{ik} are the probability rates of hopping to the left, and g_{ki} to the right. The event of exchange occurs if out of two adjacent sites one is a vacancy and the other is occupied by a particle, or each of the sites is occupied by a particle of a different type. The n-species symmetric simple exclusion process is known as the lattice gas model of particle hopping between nearest-neighbor sites with a constant rate $g_{ik} = g_{ki} = g$. The n-species asymmetric simple exclusion process with hopping in a preferred direction is the diffusion-driven lattice gas of particles moving under the action of an external field. The process is totally asymmetric if all jumps occur in one direction only, and partially asymmetric if there is a different non-zero probability of both left and right hopping. The number of particles n_i of each species in the bulk is conserved and this is the case of periodic boundary conditions. In the case of open systems, the lattice gas is coupled to external reservoirs of particles of fixed density. In most studied examples one considers phase transitions inducing boundary processes when a particle of type k, $k = 1, 2, \ldots, n-1$ is added with a rate L_k^0 and/or removed with a rate L_0^k at the left end of the chain, and it is removed with a rate R_0^k and/or added with a rate R_k^0 at the right end of the chain.

In the matrix-product states approach the boundary rate matrices define the boundary vectors with respect to which the stationary probability distribution is related to an expectation value of product of matrices obeying the quadratic algebra (4). The problem to be solved is to find matrix representations of the quadratic algebra consistent with the boundary conditions (7), namely that the combinations $(L_i^k D_k + x_i)$ and $(R_i^k D_k - x_i)$ have common vectors with eigenvalue zero, where the only non-vanishing boundary rates are $L_0^k, L_k^0, R_0^k, R_k^0$, $k = 1, 2, \ldots, n-1$. Once this problem is solved important physical quantities like correlation functions, currents, density profiles can be obtained which is the advantage of the matrix-product states approach. Despite the extensive study of simple generalizations of the exclusion process solutions of systems of n-species is lacking.

We are implementing here the deformed squeezed states introduced and studied in the previous section and the deformed coherent states to obtain a solution to the general n boundary value problem.

4.1 Deformed squeezed and coherent state solution of the boundary problem for the n-species process

The algebra for the n-species open asymmetric exclusion process of a diffusion system coupled at both boundaries to external reservoirs of particles of fixed density has the form

$$D_{n-1}D_0 - qD_0D_{n-1} = \frac{x_0}{g_{n-1,0}}D_{n-1} - \frac{x_{n-1}}{g_{n-1,0}}D_0,$$

$$D_0D_k - q_kD_kD_0 = -\frac{x_0}{g_k}D_k, \tag{63}$$
$$D_kD_{n-1} - q_kD_{n-1}D_k = \frac{x_{n-1}}{g_k},$$
$$D_kD_l - q_{kl}^{-1}D_lD_k = 0,$$

where $k, l = 1, 2, \ldots, n-2$, $x_0 + x_{n-1} = 0$ and

$$q = \frac{g_{0,n-1}}{g_{n-1,0}}, \qquad q_{kl} = \frac{g_{kl}}{g_{lk}}, \qquad q_k = \frac{g_{k0}}{g_{0k}} = \frac{g_{n-1,k}}{g_{k,n-1}}. \tag{64}$$

The equalities in the last formula, together with the relations

$$g_k = g_{0k} = g_{k,n-1},$$
$$g_{0k} - g_{k0} = g_{k,n-1} - g_{n-1,k} = g_{0,n-1} - g_{n-1,0}, \tag{65}$$

yield a mapping to the commutation relations of a q-deformed Heisenberg algebra (see (10)) of $n-1$ oscillators $a_k, a_k^+, k = 0, 1, 2, \ldots, n-2$. A solution is obtained by a shift of the oscillators a_0, a_0^+

$$D_0 = \frac{x_0}{g_{n-1,0}}\left(\frac{1}{1-q} + \frac{a_0^+}{\sqrt{1-q}}\right), \tag{66}$$
$$D_{n-1} = \frac{-x_{n-1}}{g_{n-1,0}}\left(\frac{1}{1-q} + \frac{a}{\sqrt{1-q}}\right),$$

and by the identification of the rest of the generators D_k, $k = 1, 2, \ldots, n-2$ with the remaining $n-2$ creation operators a_k^+:

$$D_k = a_k^+, \qquad k \neq 0. \tag{67}$$

For the phase transition inducing boundary processes, when a particle of type k is added with a rate L_k^0 and removed with a rate L_0^k at the left end of the chain and when it is removed with a rate R_0^k and added with a rate R_k^0 at the right end of the chain, the boundary vectors are defined by the systems of equations

$$\langle w|((-L_1^0 - L_2^0 - \ldots - L_{n-1}^0)D_0$$
$$+ L_0^1D_1 + L_0^2D_2 + \ldots + L_0^{n-1}D_{n-1} + x_0) = 0, \tag{68}$$
$$\langle w|(L_1^0D_0 - L_0^1D_1) = 0,$$
$$\langle w|(L_2^0D_0 - L_0^2D_2) = 0,$$
$$\vdots$$
$$\langle w|(L_{n-2}^0D_0 - L_0^{n-2}D_{n-2}) = 0,$$
$$\langle w|(L_{n-1}^0D_0 - L_0^{n-1}D_{n-1} + x_{n-1}) = 0,$$

and

$$((-R_1^0 - R_2^0 - \ldots - R_{n-1}^0)D_0 + R_0^1D_1$$
$$+ R_0^2D_2 + \ldots + R_0^{n-1}D_{n-1} - x_0)|v\rangle = 0, \tag{69}$$
$$(R_1^0D_0 - R_0^1D_1)|v\rangle = 0,$$
$$(R_2^0D_0 - R_0^2D_2)|v\rangle = 0,$$
$$\vdots$$
$$(R_{n-2}^0D_0 - R_0^{n-2}D_{n-2})|v\rangle = 0,$$
$$(R_{n-1}^0D_0 - R_0^{n-1}D_{n-1} - x_{n-1})|v\rangle = 0.$$

The two systems are similar and can be solved by the same procedure. From the second to the last but one equation in (68) and (69), one has

$$\langle w|L_0^kD_k = \langle w|L_k^0D_0, \tag{70}$$
$$R_0^kD_k|v\rangle = R_k^0D_0|v\rangle, \tag{71}$$

for $k = 1, 2, \ldots n-2$. Hence one inserts (70) in the first equation of the system (68) and (71) in the first equation of the system (69) to obtain in both cases an equation that coincides with the last equation of the correspondings systems.

Thus the system for the left and right boundary vectors are reduced to the pair of equations

$$\langle w|(L_{n-1}^0D_0 - L_0^{n-1}D_{n-1}) = \langle w|, \tag{72}$$
$$(R_0^{n-1}D_{n-1} - R_{n-1}^0D_0)|v\rangle = |v\rangle.$$

Making use of the explicit solution for D_{n-1} and D_0 as shifted deformed oscillators (with $x_0 = -x_1 = 1$), we rewrite (72) as

$$(R_0^{n-1}a_0 - R_{n-1}^0a_0^+)|v\rangle$$
$$= \sqrt{1-q}\left(g_{n-1,0} - \frac{R_0^{n-1} - R_{n-1}^0}{1-q}\right)|v\rangle, \tag{73}$$
$$\langle w|(L_{n-1}^0a_0^+ - L_0^{n-1}a_0)$$
$$= \langle w|\left(g_{n-1,0} - \frac{L_{n-1}^0 - L_0^{n-1}}{1-q}\right)\sqrt{1-q}.$$

The latter equations, in accordance with (35), determine the boundary vectors as squeezed coherent states of the deformed boson operators a_0, a_0^+ corresponding to the eigenvalues

$$v = \sqrt{1-q}\left(g_{n-1,0} - \frac{R_0^{n-1} - R_{n-1}^0}{1-q}\right), \tag{74}$$
$$w = \sqrt{1-q}\left(g_{n-1,0} - \frac{L_{n-1}^0 - L_0^{n-1}}{1-q}\right).$$

The explicit form of these vectors is readily written, namely

$$\langle w| = \langle n|\sum_{n=0}^{\infty}\frac{w^n}{\sqrt{[n]!}}e_q^{-\frac{1}{2}vw} \text{ and } |v\rangle = e_q^{-\frac{1}{2}vw}\sum_{n=0}^{\infty}\frac{v^n}{\sqrt{[n]!}}|n\rangle.$$

We therefore conclude to the following.

The left and right boundary vectors are squeezed coherent states of the shifted deformed annihilation and creation operators D_{n-1} and D_0, associated with the non-zero boundary parameters x_{n-1} and x_0, and with eigenvalues depending on the right and left boundary rates:

$$(R_0^{n-1}a_0 - R_{n-1}^0 a_0^+)|v\rangle = A|v\rangle = v|v\rangle, \tag{75}$$
$$\langle w|(L_{n-1}^0 a_0^+ - L_0^{n-1}a) = \langle w|A^+ = \langle w|w,$$

where the eigenvalues v and w are given by (74).

The operators A and A^+ satisfy the same deformed commutation relation as a and a^+, as was outlined in Sect. 3, with the only difference that they are not Hermitian conjugate. However, their conjugation property is consistent with the involution of the quadratic algebra (4) which reflects the left–right symmetry of the model. From the inverse linear maps, with $R_0^{n-1}L_{n-1}^0 - L_0^{n-1}R_{n-1}^0 \neq 0$, we obtain

$$a_0 = \frac{L_{n-1}^0}{R_0^{n-1}L_{n-1}^0 - L_0^{n-1}R_{n-1}^0}A + \frac{R_{n-1}^0}{R_0^{n-1}L_{n-1}^0 - L_0^{n-1}R_{n-1}^0}A^+, \tag{76}$$
$$a_0^+ = \frac{R_0^{n-1}}{R_0^{n-1}L_{n-1}^0 - L_0^{n-1}R_{n-1}^0}A^+ + \frac{L_0^{n-1}}{R_0^{n-1}L_{n-1}^0 - L_0^{n-1}R_{n-1}^0}A,$$

with the help of which the mean values of the generators D_0, D_{n-1} and the rest D_k for $k = 1, 2, \ldots, n-2$ are readily found:

$$\langle w|D_0|v\rangle = \frac{1}{g_{n-1,0}(R_0^{n-1}L_{n-1}^0 - L_0^{n-1}R_{n-1}^0)} \times \left(\frac{1}{1-q} + \frac{R_0^{n-1}w + L_0^{n-1}v}{\sqrt{1-q}}\right), \tag{77}$$
$$\langle w|D_{n-1}|v\rangle = \frac{1}{g_{n-1,0}(R_0^{n-1}L_{n-1}^0 - L_0^{n-1}R_{n-1}^0)} \times \left(\frac{1}{1-q} + \frac{R_{n-1}^0 w + L_{n-1}^0 v}{\sqrt{1-q}}\right),$$
$$\langle w|D_k|v\rangle = \frac{L_k^0}{L_0^k}\langle w|D_0|v\rangle = \frac{R_k^0}{R_0^k}\langle w|D_0|v\rangle.$$

With these expressions at hand, it is easy to calculate the expectation value of any monomial of the form $\langle w|D_{s_1} D_{s_2}\ldots Ds_L|v\rangle$ (where $D_{s_i} = D_j$ for $i = 1, 2, \ldots, L$, $j = 0, 1, 2, \ldots, n-1$), which enters the stationary probability distribution, the current, and the correlation functions. One first makes use of the algebra to bring all generators D_k for $k = 1, 2, \ldots, n-2$ to the very right or to the very left, which results in an expression of the expectation value as a power in D_0 and D_{n-1}. Then one writes the arbitrary power of D_0, D_{n-1} as a normally ordered product of A and A^+ to obtain, upon using the eigenvalue properties of the latter, an expression for the relevant physical quantity in terms of the probability-rate-dependent boundary eigenvalues v and w.

We note that if the boundary processes are such that there are only incoming particles of $(n-1)$th-type at the left boundary and only outgoing $(n-1)$th-type particles at the right boundary, i.e. $L_0^{n-1} = R_{n-1}^0 = 0$ in (75), then the eigenstate equations define the boundary vectors $|v\rangle$ and $\langle w|$ as q-deformed coherent states. Using the eigenvalue properties of the latter one can likewise obtain the physical quantities of interest for the system. The value $q \neq 0$ corresponds to a partially asymmetric and $q = 0$ to a totally asymmetric diffusion in the bulk of the $n-1$-type particle. The deformed oscillator coherent states defined for $0 < q < 1$ and for $q = 0$ provide a unified description of both the partially and the totally asymmetric hopping of a given type of particle.

4.2 Example: The two-species model with incoming and outgoing particles at both boundaries

As an example we consider the two-species partially asymmetric simple exclusion process. We simplify the notation, namely at the left boundary a particle can be added with probability αdt and removed with probability γdt, and at the right boundary it can be removed with probability βdt and added with probability δdt. The system is described by the configuration set $s_1, s_2, \ldots, s_L$ where $s_i = 0$ if a site $i = 1, 2, \ldots, L$ is empty and $s_i = 1$ if a site i is occupied by a particle. The particles hop with a probability $g_{01}dt$ to the left and with a probability $g_{10}dt$ to the right, where without loss of generality we can choose the right probability rate $g_{10} = 1$ and the left probability rate $g_{01} = q$. The quadratic algebra $D_1D_0 - qD_0D_1 = D_0 + D_1$ is solved by a pair of deformed oscillators a, a^+ (see (67) with $n = 2$). The boundary conditions have the form

$$(\beta D_1 - \delta D_0)|v\rangle = |v\rangle, \tag{78}$$
$$\langle w|(\alpha D_0 - \gamma D_1) = \langle w|.$$

For a given configuration $(s_1, s_2, \ldots, s_L)$ the stationary probability is given by the expectation value

$$P(s) = \frac{\langle w|D_{s_1}D_{s_2}\ldots D_{s_L}|v\rangle}{Z_L}, \tag{79}$$

where $D_{s_i} = D_1$ if a site $i = 1, 2, \ldots, L$ is occupied and $D_{s_i} = D_0$ if a site i is empty and $Z_L = \langle w|(D_0 + D_1)^L|v\rangle$ is the normalization factor to the stationary probability distribution. Within the matrix-product ansatz, one can also evaluate physical quantities such as the current J through a bond between site i and site $i+1$, the mean density $\langle s_i\rangle$ at a site i, the two-point correlation function

$\langle s_i s_j \rangle$:

$$J = \frac{Z_{L-1}}{Z_L}, \tag{80}$$

$$\langle s_i \rangle = \frac{\langle w|(D_0+D_1)^{i-1}D_1(D_0+D_1)^{L-i}|v\rangle}{Z_L},$$

$$\langle s_i s_j \rangle = \frac{\langle w|(D_0+D_1)^{i-1}D_1(D_0+D_1)^{j-i-1}D_1(D_0+D_1)^{L-j}|v\rangle}{Z_L},$$

and higher correlation functions.

In terms of the deformed boson operators the boundary conditions read

$$(\beta a - \delta a^+)|v\rangle = \sqrt{1-q}\left(1 - \frac{\beta-\delta}{1-q}\right)|v\rangle \tag{81}$$

$$\langle w|(\alpha a^+ - \gamma a) = \langle w|\left(1 - \frac{\alpha-\gamma}{1-q}\right)\sqrt{1-q}.$$

Hence, according to (35), the boundary vectors $|v\rangle$ and $\langle w|$ are squeezed coherent states, eigenstates of an annihilation and a creation operator A, A^+:

$$(\beta a - \delta a^+)|v\rangle = A|v\rangle = v|v\rangle, \tag{82}$$

$$\langle w|(\alpha a^+ - \gamma a) = \langle w|A^+ = \langle w|w,$$

corresponding to the eigenvalues

$$v(\beta,\delta) = \sqrt{1-q}\left(1 - \frac{\beta-\delta}{1-q}\right), \tag{83}$$

$$w(\alpha,\gamma) = \sqrt{1-q}\left(1 - \frac{\alpha-\gamma}{1-q}\right).$$

The explicit form of the (unnormalized) vectors in the oscillator Fock space representation is given by $\langle w| = \sum_{n=0}^{\infty} \frac{w^n(\alpha,\gamma)}{\sqrt{[n]!}}|n\rangle$, $|v\rangle = \sum_{n=0}^{\infty} \frac{v^n(\beta,\delta)}{\sqrt{[n]!}}|n\rangle$. As already noted the operators A and A^+ are not each other's Hermitian conjugate. To find the expectation values of normally ordered monomials in D_0 and D_1, we make use of the inverse transformation

$$a = \frac{\alpha}{\alpha\beta-\gamma\delta}A + \frac{\delta}{\alpha\beta-\gamma\delta}A^+, \tag{84}$$

$$a^+ = \frac{\beta}{\alpha\beta-\gamma\delta}A^+ + \frac{\gamma}{\alpha\beta-\gamma\delta}A.$$

Hence with $\Delta = \alpha\beta - \gamma\delta \neq 0$

$$D_0 + D_1 = \frac{2}{1-q} + \frac{\alpha+\gamma}{\Delta\sqrt{1-q}}A + \frac{\beta+\delta}{\Delta\sqrt{1-q}}A^+, \tag{85}$$

and the normalization factor $\langle w|(D_0+D_1)^L|v\rangle$ to the stationary probability distribution can easily be calculated in terms of the operators A and A^+. One has

$$(D_0+D_1)^L = \left(\frac{2}{1-q} + \frac{\alpha+\gamma}{\Delta\sqrt{1-q}}A + \frac{\beta+\delta}{\Delta\sqrt{1-q}}A^+\right)^L$$

$$= \sum_{m=0}^{L} \frac{L!}{m!(L-m)!}\frac{2^{L-m}\Delta^{-m}}{(1-q)^{L-\frac{m}{2}}} \times((\alpha+\gamma)A + (\beta+\delta)A^+)^m. \tag{86}$$

To evaluate the above expression one makes use of the eigenvalue properties of the squeezed states $\langle w|$ and $|v\rangle$. For this purpose one first applies the procedure for normal ordering of polynomials in A, A^+ outlined in Sect. 4.3. The difference with the normalization factor for boundary processes with only incoming particles at the left end and only outgoing particles at the right end evaluated there is that here one has to normally order boson operators whose deformed commutator is not normalized to unity. This results in the formula

$$((\alpha+\gamma)A + (\beta+\delta)A^+)^m$$

$$= \sum_{k=0}^{[m/2]} S_k^m (\alpha+\gamma)^k(\beta+\delta)^k \sum_{l=0}^{m-2k} \frac{[m-2k]!}{[l]![m-2k-l]!} \times((\beta+\delta)A^+)^l((\alpha+\gamma)A)^{m-2k-l} \tag{87}$$

One explores next the eigenvalue properties of the operators A, A^+ with respect to the vectors $|v\rangle$ and $\langle w|$ and finds the normalization factor $\langle w|(D_0+D_1)^L|v\rangle = Z_L$ to the stationary probability distribution:

$$Z_L = \sum_{m=0}^{L} \binom{L}{m} \frac{2^{L-m}}{(1-q)^{L-\frac{m}{2}}} \times \sum_{k=0}^{[m/2]} \sum_{l=0}^{m-2k} S_m^{(k)} \frac{(\alpha+\gamma)^k(\beta+\delta)^k}{(\alpha\beta-\gamma\delta)^m} \binom{m-2k}{l}_q \times((\beta+\delta)w)^l((\alpha+\gamma)v)^{m-2k-l}. \tag{88}$$

Consequently one directly obtains an expression for the current J. We note that an explicit formula for the normalization factor to the stationary probability distribution (and hence for the current) of the two-species diffusion system with incoming and outgoing particles at both boundaries has not been written elsewhere. Using the prescription of normal ordering, one can readily calculate the correlation functions and any other quantity of interest like density profiles, etc. Since none of the physical quantities of interest for this process have been presented elsewhere, this strongly supports the squeezed coherent state solution as a powerful method for the study of stochastic systems.

4.3 Example: The two-species asymmetric simple exclusion process, with only incoming particles at the left boundary and only outgoing particles at the right one

The model is exactly solvable through the matrix-product states approach [8, 23].

We comment on it here for the reason of just stressing the utility of the q-deformed oscillator coherent states which provide the most simple and convenient approach to an unified solution of both the partially and the totally asymmetric processes. In the partially asymmetric case the probability rate of hopping to the left is $g_{01} = q$ while the right probability rate is $g_{10} = 1$. The totally asymmetric exclusion process of particles hopping to the right only is obtained for $q = 0$. At the left boundary a particle can be added with a probability αdt and it can be removed at the right boundary with a probability βdt. The quadratic algebra is generated by an unit and two generators obeying the following relations.

Case A. The partially asymmetric simple exclusion process $(0 < q < 1)$

$$D_1 D_0 - q D_0 D_1 = D_0 + D_1. \tag{89}$$

Case B. The totally asymmetric simple exclusion process $(q = 0)$

$$D_1 D_0 = D_0 + D_1, \tag{90}$$

with the same boundary conditions defining in both cases the boundary vectors $\langle w|$ and $|v\rangle$:

$$\langle w|D_0 = \langle w|\frac{1}{\alpha}, \qquad D_1|v\rangle = \frac{1}{\beta}|v\rangle. \tag{91}$$

The algebraic solutions (with the corresponding boundary problems) for the partially and for the totally asymmetric cases are of the form of shifted deformed oscillators for a real parameter $0 < q < 1$ and for $q = 0$, respectively.

Case A.

$$D_0 = \frac{1}{1-q} + \frac{a^+}{\sqrt{1-q}}, \qquad D_1 = \frac{1}{1-q} + \frac{a}{\sqrt{1-q}}. \tag{92}$$

To solve the boundary problem we choose the vector $|v\rangle$ to be the (unnormalized!) eigenvector of the annihilation operator a for a real value of the parameter v and the vector $\langle w|$ to be the eigenvector (unnormalized and different from the conjugated one) of the creation operator for the real parameter w:

$$|v\rangle = e_q^{-\frac{1}{2}vw} e_q^{va^+}|0\rangle, \qquad \langle w| = \langle 0|e_q^{wa} e_q^{-\frac{1}{2}wv}. \tag{93}$$

The factor $e_q^{-\frac{1}{2}vw}$ in (93) is due to the condition $\langle w|v\rangle = 1$, which is a convenient choice in physical applications. According to the algebraic solution, these are also eigenvectors of the shifted operators with the corresponding relations of the eigenvalues

$$\frac{1}{\alpha} = \frac{1}{1-q} + \frac{w}{\sqrt{1-q}}, \qquad \frac{1}{\beta} = \frac{1}{1-q} + \frac{v}{\sqrt{1-q}}. \tag{94}$$

Hence the boundary vectors $|v\rangle$ and $\langle w|$ are a subset of the coherent states of the q-deformed Heisenberg algebra, labelled by the positive real parameters $v(\alpha, q)$ and $w(\beta, q)$ defined in (94). The relation of the boundary vectors to the coherent states simplifies the calculation of the stationary probability distribution. Since, according to the algebraic solution,

$$(D_0 + D_1)^L = \left(\frac{2}{1-q} + \frac{a^+ + a}{\sqrt{1-q}}\right)^L = \sum_{m=0}^{L} \frac{L!}{m!(L-m)!} \frac{2^{L-m}}{(1-q)^{L-m}(\sqrt{1-q})^m}(a^+ + a)^m, \tag{95}$$

in order to find the expectation values with respect to the coherent states, one has to normally order the mth power of the linear combination $a + a^+$, using $aa^+ - qa^+a = 1$. This is achieved with the help of the Stirling numbers

$$(a^+ + a)^m = \sum_{k=0}^{[m/2]} S_m^{(k)} \sum_{l=0}^{m-2k} \frac{[m-2k]!}{[l]![m-2k-l]!}(a^+)^l a^{m-2k-l}, \tag{96}$$

where the q-deformed Stirling numbers $S_m^{(k)}$ satisfy the recurrence relation

$$S_{m+1}^{(k)} = [k]S_m^{(k)} + S_m^{(k-1)}, \tag{97}$$

with $S_m^{(0)} = \delta_{0m}$, $S_m^{(1)} = S_m^{(m)} = 1$ and $S_m^{(m-1)} = \sum_{i=1}^{i=m-1}[i]$. For the correlation functions one also needs the expressions

$$a^k a^+ = q^k a^+ a^k + [k]a^{k-1}, \qquad a(a^+)^k = q^k (a^+)^k a + [k](a^+)^{k-1}. \tag{98}$$

Using these relations one can easily find the relevant physical quantities of the system. Thus for the normalization factor Z^L one obtains

$$\langle w|(D_0 + D_1)^L|v\rangle = \sum_{m=0}^{L} \frac{L!}{m!(L-m)!} \frac{2^{L-m}}{(1-q)^{L-\frac{m}{2}}} \times \sum_{k=0}^{[m/2]} \sum_{l=0}^{m-2k} S_m^{(k)} \frac{[m-2k]!}{[l]![m-2k-l]!} w^l v^{m-2k-l}. \tag{99}$$

It can be verified, after rescaling the parameters v and w by $\frac{1}{\sqrt{1-q}}$, that this expression coincides with the one evaluated in [23] up to the factor $\langle w|v\rangle$, which is chosen there to be $\langle w|v\rangle \neq 1$.

Case B.

$$D_0 = 1 + a^+_{q=0}, \qquad D_1 = 1 + a_{q=0}. \tag{100}$$

As the algebra itself, the solution (100) and the boundary vectors are also obtained as the limit $q \to 0$ of the q-dependent solution and eigenvectors where the representation of the oscillator operators in (100) is found from (13) with $q = 0$, namely $a^+|n\rangle = |n+1\rangle$, $a|n\rangle = |n-1\rangle$ and

$$w = \frac{1-\alpha}{\alpha}, \qquad v = \frac{1-\beta}{\beta}. \tag{101}$$

Hence the boundary vectors have the form

$$\langle w| = \langle n| \sum_{n=0}^{\infty} \left(\frac{1-\alpha}{\alpha}\right)^n \left(\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\alpha\beta}\right)^{\frac{1}{2}},$$

$$|v\rangle = \left(\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\alpha\beta}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \left(\frac{1-\beta}{\beta}\right)^n |n\rangle. \quad (102)$$

The physical quantities of the model are readily obtained from the partially asymmetric case in the limit $q \to 0$. Equation (96) becomes simply

$$(a + a^+)^L\big|_{q=0} = \sum_{k=0}^{[m/2]} S_m^{(k)}\big|_{q=0} \sum_{l=0}^{m-2k} (a^+_{q=0})^l (a_{q=0})^{m-2k-l}, \quad (103)$$

where now

$$S_{m+1}^{(k)}\big|_{q=0} = S_m^{(k)}\big|_{q=0} + S_m^{(k-1)}\big|_{q=0} \text{ and } S_m^{(m-1)}\big|_{q=0} = m-1.$$

The expression for Z_L becomes

$$\langle w|(D_0 + D_1)^L|v\rangle \quad (104)$$
$$= \sum_{m=0}^{L} \frac{2^{l-m} L!}{m!(L-m)!} \sum_{k=0}^{[m/2]} \sum_{l=0}^{m-2k} S_m^{(k)}|_{q=0} w^l v^{m-2k-l}.$$

Inserting in (104) the expressions for v and w in terms of α and β from (101), it can be verified, after some algebra, that it coincides with the expression for the normalization factor obtained in [8] (as the current and the correlation functions do coincide too). The coherent-state description thus provides an unified solution of the partially and fully asymmetric simple exclusion models.

To summarize, we have suggested and studied deformed boson oscillator squeezed coherent states as eigenstates of linear combinations of deformed annihilation and creation operators. The analyses of their properties show that they exibit squeezing properties like the canonical squeezed states and can thus be considered as the q-analogues of the canonical harmonic oscillator squeezed states. We have applied the q-deformed squeezed and coherent states to obtain within the matrix-product states approach a boundary problem solution to a multiparticle (general n) open stochastic system of lattice Brownian motion. We have shown that depending on the boundary processes, the boundary vectors are either deformed coherent or deformed squeezed states of the deformed oscillator algebra used for the solution. The coherent states provide an unified description of both the partially and the fully asymmetric cases, the solution of the fully asymmetric one being obtained in the limit $q \to 0$ of the deformation parameter q. The discussed deformed squeezed- and coherent-state solution of the boundary problem for the n-species stochastic diffusion process is proposed as a generalization of the known examples within the matrix-product states approach. We emphasize that in the two-species example solved by means of the deformed squeezed states the expression for the normalization factor to the stationary probability distribution (and hence for the current) have not been presented elswhere; this strongly supports the squeezed coherent boundary solution as a convenient method for studying the stochastic diffusion systems. In applying the deformed squeezed states, however, we have utilized their eigenvalue properties only and not explored their squeezing properties. The squeezing condition relates all the parameters involved in the processess and in our opinion, it is worth studying the consequences of squeezing on the stochastic dynamics.

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